M1

Time: 3 Hours

Number of Questions: 30 **Max Marks: 100**

INSTRUCTIONS

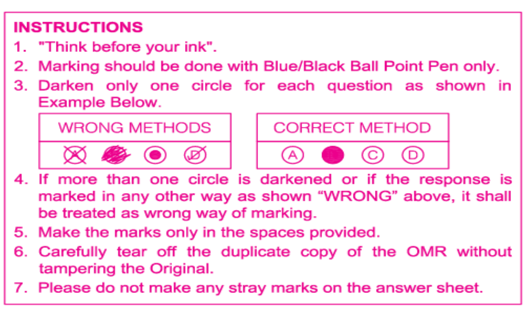
1. Use of mobile phones, smartphones, iPads, calculators, programmable wrist watches is **STRICTLY PROHIBITED.** Only ordinary pens and pencils are allowed inside the examination hall.

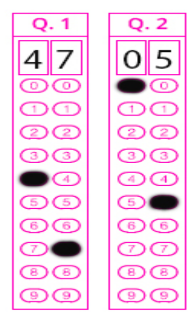
2. The correction is done by machines through scanning. On the OMR Sheet, darken bubbles completely with a **black or blue ball pen**. Please **DO NOT use a pencil or a gel pen**. Darken the bubbles completely, only after you are sure of your answer; else, erasing may lead to the OMR sheet getting damaged and the machine may not be able to read the answer. .

3. The name, email address, and date of birth entered on the OMR sheet will be your login credentials for accessing your score.

4. Incompletely, incorrectly or carelessly filled information may disqualify your candidature.

5. Each question has a one or two digit number as answer. The first diagram below shows improper and proper way of darkening the bubbles with detailed instructions. The second diagram shows how to mark a 2-digit number and a 1-digit number.





6. The answer you write on OMR sheet is irrelevant. The darkened bubble will be considered as your final answer.

7. Questions 1 to 10 carry 2 marks each; questions 11 to 20 carry 3 marks each; questions 21 and 30 carry 5 marks each.

8. All questions are compulsory.

9. There are no negative marks.

10. Do all rough work in the space provided below for it. You also have blank pages at the end of the question paper to continue with rough work.

11. After the exam, you may take away the Candidate's copy of the OMR sheet.

12. Preserve your copy of OMR sheet till the end of current olympiad season. You will need it later for verification purposes.

13. You may take away the question paper after the examination.

Note:

1. $\mathbb{N}$ denotes the set of all natural numbers, $1, 2, 3, \ldots$.
2. For a positive real number x, $\sqrt{x}$ denotes the positive square root of x. For example, $\sqrt{4} = +2$.
3. Unless otherwise specified, all numbers are written in base 10.

Questions

1. Let n be a positive integer such that $1 \leq n \leq 1000$. Let M_n be the number of integers in the set $X_n = \{\sqrt{4n+1}, \sqrt{4n+2}, \ldots, \sqrt{4n+1000}\}$. Let

$$a = \max\{M_n : 1 \leq n \leq 1000\}, \text{ and } b = \min\{M_n : 1 \leq n \leq 1000\}.$$

Find $a - b$.

2. Find the number of elements in the set

$$\{(a, b) \in \mathbb{N} : 2 \leq a, b \leq 2023, \log_a(b) + 6 \log_b(a) = 5\}$$

3. Let α and β be positive integers such that

$$\frac{16}{37} < \frac{\alpha}{\beta} < \frac{7}{16}.$$

Find the smallest possible value of β.

4. Let x, y be positive integers such that

$$x^4 = (x-1)(y^3 - 23) - 1.$$

Find the maximum possible value of $x + y$.

SPACE FOR ROUGH WORK

5. In a triangle ABC, let E be the midpoint of AC and F be the midpoint of AB. The medians BE and CF intersect at G. Let Y and Z be the midpoints of BE and CF respectively. If the area of triangle ABC is 480, find the area of triangle GYZ.

6. Let X be the set of all even positive integers n such that the measure of the angle of some regular polygon is n degrees. Find the number of elements in X.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is colored either red or yellow with opposite faces always of the same color. Two dice are considered to have the same design if one of them can be rotated to obtain a dice that has the same numbers and colors on the corresponding faces as the other one. Find the number of distinct dice that can be designed.

8. Given a 2×2 tile and seven dominoes (2×1 tile), find the number of ways of tiling (that is, cover without leaving gaps and without overlapping of any two tiles) a 2×7 rectangle using some of these tiles.

SPACE FOR ROUGH WORK

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

10. The sequence $\langle a_n \rangle_{n \geq 0}$ is defined by $a_0 = 1, a_1 = -4$ and $a_{n+2} = -4a_{n+1} - 7a_n$, for $n \geq 0$. Find the number of positive integer divisors of $a_{50}^2 - a_{49}a_{51}$.

11. A positive integer m has the property that m^2 is expressible in the form $4n^2 - 5n + 16$ where n is an integer (of any sign). Find the maximum possible value of $|m - n|$.

12. Let $P(x) = x^3 + ax^2 + bx + c$ be a polynomial where a, b, c are integers and c is odd. Let p_i be the value of $P(x)$ at $x = i$. Given that $p_1^3 + p_2^3 + p_3^3 = 3p_1p_2p_3$, find the value of $p_2 + 2p_1 - 3p_0$.

SPACE FOR ROUGH WORK

13. The ex-radii of a triangle are $10\frac{1}{2}, 12$ and 14. If the sides of the triangle are the roots of the cubic $x^3 - px^2 + qx - r = 0$, where p, q, r are integers, find the integer nearest to $\sqrt{p+q+r}$.

14. Let ABC be a triangle in the xy plane, where B is at the origin $(0,0)$. Let BC be produced to D such that $BC : CD = 1 : 1$, CA be produced to E such that $CA : AE = 1 : 2$ and AB be produced to F such that $AB : BF = 1 : 3$. Let $G(32, 24)$ be the centroid of the triangle ABC and K be the centroid of the triangle DEF. Find the length GK.

15. Let $ABCD$ be a unit square. Suppose M and N are points on BC and CD respectively such that the perimeter of triangle MCN is 2. Let O be the circumcentre of triangle MAN, and P be the circumcentre of triangle MON. If $\left(\frac{OP}{OA}\right)^2 = \frac{m}{n}$ for some relatively prime positive integers m and n, find the value of $m+n$.

16. The six sides of a convex hexagon $A_1A_2A_3A_4A_5A_6$ are colored red. Each of the diagonals of the hexagon is colored either red or blue. If N is the number of colorings such that every triangle $A_iA_jA_k$, where $1 \le i < j < k \le 6$, has at least one red side, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

17. Consider the set

$$\mathcal{S} = \{(a, b, c, d, e) : 0 < a < b < c < d < e < 100\}$$

where a, b, c, d, e are integers. If D is the average value of the fourth element of such a tuple in the set, taken over all the elements of $\mathcal{S}$, find the largest integer less than or equal to D.

18. Let $\mathcal{P}$ be a convex polygon with 50 vertices. A set $\mathcal{F}$ of diagonals of $\mathcal{P}$ is said to be *minimally friendly* if any diagonal $d \in \mathcal{F}$ intersects at most one other diagonal in $\mathcal{F}$ at a point interior to $\mathcal{P}$. Find the largest possible number of elements in a minimally friendly set $\mathcal{F}$.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

SPACE FOR ROUGH WORK

20. For any finite non empty set X of integers, let $\max(X)$ denote the largest element of X and $|X|$ denote the number of elements in X. If N is the number of ordered pairs (A, B) of finite non-empty sets of positive integers, such that

$$\max(A) \times |B| = 12; \text{ and}$$
$$|A| \times \max(B) = 11$$

and N can be written as $100a + b$ where a, b are positive integers less than 100, find $a + b$.

21. For $n \in \mathbb{N}$, consider non-negative integer-valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

22. In an equilateral triangle of side length 6, pegs are placed at the vertices and also evenly along each side at a distance of 1 from each other. Four distinct pegs are chosen from the 15 interior pegs on the sides (that is, the chosen ones are not vertices of the triangle) and each peg is joined to the respective opposite vertex by a line segment. If N denotes the number of ways we can choose the pegs such that the drawn line segments divide the interior of the triangle into exactly nine regions, find the sum of the squares of the digits of N.

SPACE FOR ROUGH WORK

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

24. A trapezium in the plane is a quadrilateral in which a pair of opposite sides are parallel. A trapezium is said to be non-degenerate if it has positive area. Find the number of mutually non-congruent, non-degenerate trapeziums whose sides are four distinct integers from the set $\{5, 6, 7, 8, 9, 10\}$.

25. Find the least positive integer n such that there are at least 1000 unordered pairs of diagonals in a regular polygon with n vertices that intersect at a right angle in the interior of the polygon.

SPACE FOR ROUGH WORK

26. In the land of Binary, the unit of currency is called Ben and currency notes are available in denominations $1, 2, 2^2, 2^3, \ldots$ Bens. The rules of the Government of Binary stipulate that one can not use more than two notes of any one denomination in any transaction. For example, one can give a change for 2 Bens in two ways: 2 one Ben notes or 1 two Ben note. For 5 Ben one can give 1 one Ben note and 1 four Ben note or 1 one Ben note and 2 two Ben notes. Using 5 one Ben notes or 3 one Ben notes and 1 two Ben notes for a 5 Ben transaction is prohibited. Find the number of ways in which one can give change for 100 Bens, following the rules of the Government.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \leq a < b < d < c \leq 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

28. On each side of an equilateral triangle with side length n units, where n is an integer, $1 \leq n \leq 100$, consider $n - 1$ points that divide the side into n equal segments. Through these points, draw lines parallel to the sides of the triangle, obtaining a net of equilateral triangles of side length one unit. On each of the vertices of these small triangles, place a coin head up. Two coins are said to be *adjacent* if the distance between them is 1 unit. A *move* consists of flipping over any three mutually adjacent coins. Find the number of values of n for which it is possible to turn all coins tail up after a finite number of moves.

SPACE FOR ROUGH WORK

29. A positive integer $n > 1$ is called *beautiful* if n can be written in one and only one way as $n = a_1 + a_2 + \cdots + a_k = a_1 \cdot a_2 \cdots a_k$ for some positive integers $a_1, a_2, \ldots, a_k$, where $k > 1$ and $a_1 \geq a_2 \geq \cdots \geq a_k$. (For example 6 is beautiful since $6 = 3 \cdot 2 \cdot 1 = 3 + 2 + 1$, and this is unique. But 8 is not beautiful since $8 = 4 + 2 + 1 + 1 = 4 \cdot 2 \cdot 1 \cdot 1$ as well as $8 = 2 + 2 + 2 + 1 + 1 = 2 \cdot 2 \cdot 2 \cdot 1 \cdot 1$, so uniqueness is lost.) Find the largest beautiful number less than 100.

30. Let $d(m)$ denote the number of positive integer divisors of a positive integer m. If r is the number of integers $n \leq 2023$ for which $\sum_{i=1}^{n} d(i)$ is odd, find the sum of the digits of r.

SPACE FOR ROUGH WORK

Answers

QNo	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
Answer	22	54	23	07	10	16	48	59	17	51	14	18	58	40	03
QNo	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30
Answer	94	66	71	92	43	15	77	18	31	28	19	91	67	95	18

Clarifications for Students' Queries

Queries were received on questions 7, 9, 19, 21, 23 and 27. We have given below detailed clarifications to these queries. Some other queries on correctness of answers for a few other questions were received. We confirm that the answers given in our web-site are correct. The students who have such concerns can re-check their solutions themselves.

7. Unconventional dice are to be designed such that the six faces are marked with numbers from 1 to 6 with 1 and 2 appearing on opposite faces. Further, each face is coloured either red or yellow with opposite faces always of the same colour. The dice are distinguished by the colour pattern as well as the numbering pattern. Find the number of distinct dice that can be designed?

   The solutions submitted claiming that the answer must be 24 are incorrect.

   Since 1 and 2 are always opposite, the arrangement of 3,4,5,6 decides the number pattern. Each such arrangement corresponds to a circular permutation of 3,4,5,6. Hence there are $3! = 6$ such arrangements. Also, each pair of opposite faces have the same colour and there are 2 colours for use. Thus, the number of distinct designs is $6 \times 2^3 = 48$.

9. Find the number of triples (a, b, c) of positive integers such that

   (a) ab is a prime;

   (b) bc is a product of two primes;

   (c) abc is not divisible by square of any prime and

   (d) $abc \leq 30$.

   The query was that whether the triples are ordered or not is not specified. But we are counting triples (a, b, c) that satisfy the conditions and the notation clearly means that the triples are ordered.

   The possible triples are

$$
\begin{array}{ccccc}
(1,2,3) & (1,2,5) & (1,2,7) & (1,2,11) & (1,2,13) \\
(1,3,2) & (1,3,5) & (1,3,7) & (1,5,2) & (1,5,3) \\
(2,1,15) & (3,1,10) & (5,1,6) & (1,7,2) & (1,7,3) \\
(1,11,2) & (1,13,2) & & &
\end{array}
$$

   We have a total of 17 triples.

19. For $n \in \mathbb{N}$, let $P(n)$ denote the product of the digits in n and $S(n)$ denote the sum of the digits in n. Consider the set

$$A = \{n \in \mathbb{N} : P(n) \text{ is non-zero, square free and } S(n) \text{ is a proper divisor of } P(n)\}.$$

Find the maximum possible number of digits of the numbers in A.

The query was that the term square-free is not understood. But square-free is a standard term in Number Theory. A number is square-free if it is not divisible by the square of any prime number.

21. For $n \in \mathbb{N}$, consider non-negative integer valued functions f on $\{1, 2, \ldots, n\}$ satisfying $f(i) \geq f(j)$ for $i > j$ and $\sum_{i=1}^{n} (i + f(i)) = 2023$. Choose n such that $\sum_{i=1}^{n} f(i)$ is the least. How many such functions exist in that case?

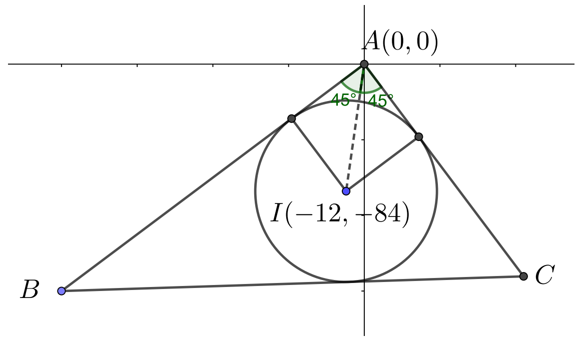
More than 90% of the queries were pertaining to this question claiming that there is ambiguity in the statement. It was argued that a function f *defined on* S meant that the domain and co-domain are both S. We have not seen such a definition of function anywhere. Also, surprisingly, almost all the mails on this question had the same text in the body of the mail.

However, there is **NO AMBIGUITY** in the statement. We are considering functions that are non-negative integer valued – hence functions that take values in non-negative integers (so that the co-domain is the set $\{0, 1, 2, \ldots\}$). The function is defined on $\{1, 2, \ldots, n\}$, where $n \in \mathbb{N}$. Thus the domain of the function is the set $\{1, 2, \ldots, n\}$. When we say a function is defined **on** a set S, by standard convention, this means that the domain of the function is S. Note that it need not take values in the same set. When we say that a function **takes values in a set** T, then by standard convention, it means that the codomain of the function is the set T.

23. In the coordinate plane, a point is called a *lattice point* if both of its coordinates are integers. Let A be the point $(12, 84)$. Find the number of right angled triangles ABC in the coordinate plane where B and C are lattice points, having a right angle at the vertex A and whose incenter is at the origin $(0, 0)$.

It was claimed that the answer should be 24. But the solution presented by the students were erroneous. The correct solution is given below.

We will shift the vertex A to the origin. We need to find the number of triangles with vertices B, C at lattice points and incenter at $(-12, -84)$. The slope of AI



is $84/12 = 7$. If m is the slope of AC, then, since $\angle IAC = 45°$, we have

$$\tan 45° = \frac{m-7}{1+7m} \Rightarrow m = -\frac{4}{3}$$

Since $AB \perp AC$, slope of AB is $\frac{3}{4}$. Hence we can write B as $(-4t, -3t)$ and C as $(3t', -4t')$ for some positive t, t'. Since B and C are lattice points, it follows that t, t' are integers.

The inradius of the triangle ABC is $r = AI \cos 45° = 60$. Also, we have $BC = AB + AC - 2r$. Thus

$$\begin{aligned} 5\sqrt{t^2 + t'^2} &= 5t + 5t' - 120 \\ \Rightarrow tt' - 24(t + t') + 288 &= 0 \\ \Rightarrow (t-24)(t'-24) &= 288 = 2^5 \cdot 3^2 \end{aligned}$$

Thus $t - 24$ can be any divisor of 288 and there are 18 divisors for 288.

The only possible negative values for $t - 24$ and $t' - 24$ are $(-16, -18)$ and $(-18, -16)$. It is easy to see that the resulting triangle for these choices do not contain $(-12, -84)$ as in-center. Also, such a choice violates the condition $BC = AB + AC - 2r$. Thus we need to count only the positive factors of 288. There are 18 divisors for 288 and hence there are 18 such triangles.

27. A quadruple (a, b, c, d) of distinct integers is said to be *balanced* if $a + c = b + d$. Let $\mathcal{S}$ be any set of quadruples (a, b, c, d) where $1 \le a < b < d < c \le 20$ and where the cardinality of S is 4411. Find the least number of balanced quadruples in $\mathcal{S}$.

It was claimed that the term cardinality was not defined. However, we maintain that cardinality of a set is a common term defined in set theory. Even the dictionary definition is "number of members in a set or a group".