FIITJEE

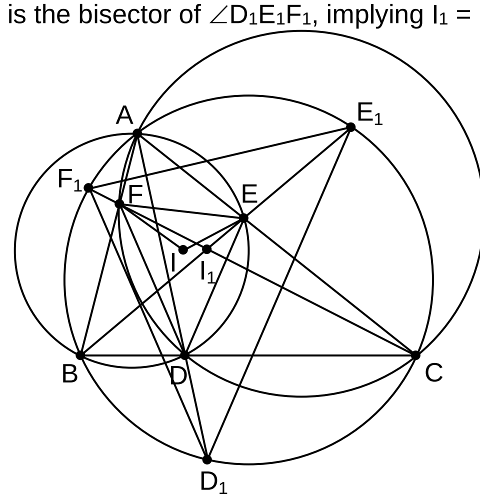
Mathematics IOQM Solutions (2021-22)

PART – B-(INMO) 2021-22

Time: 2.5 hours

Instructions:
- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks : 51
- No marks will be awarded for stating an answer without justification.
- Answer all the questions.
- PLASE READ THE INSTRUCTIONS ON THE ANSWER BOOKLET VERY CAREFULLY BEFORE ANSWERING THE QUESTIONS.

1. Note that $\angle CF_1D_1 = \angle CAD_1 = \angle EAD = \angle EBD = \angle E_1BC = \angle E_1F_1C$, so F_1C is the bisector of $\angle D_1E_1F_1$. Similarly, E_1B is the bisector of $\angle D_1E_1F_1$, implying $I_1 = BE_1 \cap CF_1$.



Now, $\angle EDF = \angle EDA + \angle FDA = \angle EBA + \angle FCA$

$= \angle E_1BA + \angle F_1CA = \angle E_1D_1A + \angle F_1D_1A = \angle E_1D_1F_1$

Therefore, $\angle EIF = 90° + \frac{1}{2}\angle EDF = 90° + \frac{1}{2}\angle E_1D_1F_1 = \angle E_1I_1F_1 = \angle EI_1F$, which proves the required concyclicity.

2. Suppose that $n \equiv 1 \pmod 3$ and σ a permutation of 1, 2, ..., n. Then

$$\sum_{i=1}^{n}\sigma(i)(-2)^{i-1} \equiv \sum_{i=1}^{n}\sigma(i) = \frac{n(n+1)}{2} \pmod 3,$$

and hence the left-hand side is non-zero.

We now show by induction that if $n \equiv 0$ or 2 (mod 3) then there exists a permutation of 1, 2, ..., n satisfying the given condition.

If n = 2 then the permutation given by $\sigma(1) = 2$, $\sigma(2) = 1$ satisfies the given condition. Similarly, if n = 3 then the permutation $\sigma(1) = 2$, $\sigma(2) = 3$, $\sigma(3) = 1$ satisfies the given condition.

Suppose that for n = m there exists a permutation σ satisfying the given condition. We consider the permutation τ of 1, 2, ..., m + 3 given by $\tau(1) = 2$, $\tau(2) = 3$, $\tau(m + 3) = 1$ and $\tau(i) = \sigma(i - 2) + 3$ for i = 3, 4, ..., m + 2. Then

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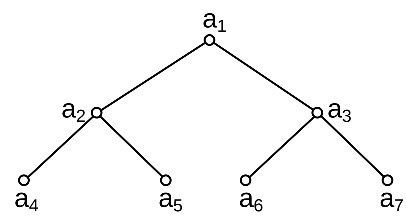
$$\sum_{i=1}^{m+3} \tau(i)(-2)^{i-1} = 2 - 6 + (-2)^{m+2} + \sum_{i=3}^{m+2} 3\cdot(-2)^{i-1}$$

$$= 2 - 6 + (-2)^{m+2} - 4\cdot((-2)^m - 1) = 0$$

Thus, by induction it follows that for every $n \equiv 0$ or 2 (mod 3) there exist a permutation satisfying the given condition.

3. (a) Given an arrangement $a_1, a_2, \ldots, a_7$ satisfying the given conditions, we can build a binary tree with nodes as in the figure below. At each node, the root node is greater than the child nodes.



Conversely, any such tree gives a valid arrangement. Observing that the root of the tree must contain the maximum of the numbers, we can choose 3 out of the other 6 numbers in $\binom{6}{3}$ ways and build the left tree and the right tree, each in 2 ways and hence the number of such trees is $2\cdot 2\cdot\binom{6}{3} = 80$.

(b) Observe that T(N) is also the number of ways of arranging any N distinct numbers into a sequence $a_1, a_2, \ldots, a_N$ satisfying the given conditions. Also, the given conditions imply that a_1 = maximum of the numbers. Now, leaving out the maximum, the rest of the $2^n - 2$ numbers can be split into two groups of $2^{n-1} - 1$ numbers each and these can be individually put into a sequences $b_1, b_2, \ldots, b_{2^{n-1}-1}$ and $c_1, c_2, \ldots, c_{2^{n-1}-1}$ satisfying the conditions in T(n – 1) ways each. Now, the required arrangement of the original given sequence can be obtained as follows:

$$a_1, b_1, c_1, b_2, b_3, c_2, c_3, b_4, b_5, b_6, b_7, c_4, c_5, c_6, c_7, \ldots$$

This gives

$$T(2^n - 1) = T(2^{n-1} - 1)^2 \binom{2^n - 2}{2^{n-1} - 1} \qquad \ldots\ (1)$$

We find the highest power of 2 that divides $\binom{2^n - 2}{2^{n-1} - 1}$:

We have $2^{n-2}\binom{2^n}{2^{n-1}} = 2^{n-2}\cdot\dfrac{2^n!}{2^{n-1}!\,2^{n-1}!} = 2^{n-2}\cdot\dfrac{2^n(2^n-1)(2^n-2)!}{2^{n-1}(2^{n-1}-1)!\,2^{n-1}(2^{n-1}-1)!} = (2^n - 1)\binom{2^n - 2}{2^{n-1} - 1}$

Now, the highest power of 2 that divides $\binom{2^n}{2^{n-1}}$ is

$$(2^{n-1} + 2^{n-2} + \ldots + 1) - 2(2^{n-2} + 2^{n-3} + \ldots + 1) = 1$$

Hence the highest power of 2 that divides $\binom{2^n - 2}{2^{n-1} - 1}$ is n – 1.

From the recurrence (1), if t_n is the highest power of 2 dividing $T(2^n - 1)$, then $t_n = 2t_{n-1} + n - 1$. From the initial conditions, $t_1 = 0$, $t_2 = 1$, $t_3 = 4$, we obtain, by an easy induction, that $t_n = 2^n - n - 1$.

(c) Suppose that $N = 2^n + 1$. It is easy to see that

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$$T(2^n+1) = T(2^{n-1}-1)T(2^{n-1}+1)\binom{2^n}{2^{n-1}+1}$$

The highest power of 2 dividing $\binom{2^n}{2^{n-1}+1}$ is n: $(2^{n-1}+1)\binom{2^n}{2^{n-1}+1} = \binom{2^n}{2^{n-1}} \cdot 2^{n-1}$

Since the highest power of 2 dividing $\binom{2^n}{2^{n-1}}$ is 1, it follows that the highest power of 2 dividing $\binom{2^n}{2^{n-1}+1}$ is n. Thus, if s_n denotes the highest power of 2 dividing $T(2^n+1)$, then

$$s_n = s_{n-1} + 2^{n-1} - (n-1) - 1 + n = s_{n-1} + 2^{n-1}$$

Hence, $s_n - s_1 = 2^n - 2$ and since $s_1 = T(3) = 2$, it follows that the highest power of 2 dividing $T(2^n + 1)$ is 2^n.

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