Mathematics IOQM Solutions (2021)

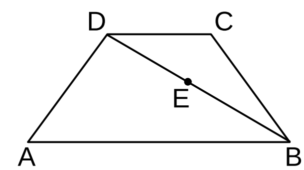
1. Let ABCD be a trapezium in which AB || CD and AB = 3CD. Let E be the midpoint of diagonal BD. If [ABCD] = n × [CDE], what is the value of n? [Here [T] denotes the area of the geometrical figure T).

Sol. **8**

[ACD] = 2[CDE] = $\frac{1}{3}[ABC]$

[ABCD] = [ACD] + [ABC]

= 2[CDE] + 6[CDE] = 8[CDE]



2. A number N in base 10, is 503 in base b and 305 in base b + 2. What is the product of the digits of N?

Sol. **64**

N = 3b + 5b^2 = 5(b + 2) + 3(b + 2)^2

$\Rightarrow$ b^2 – 6b – 7 = 0, (b – 7)(b + 1) = 0

$\Rightarrow$ b = 7

N = 248, product of digits = 64

3. If $\sum_{k=1}^{N}\frac{2k+1}{\left(k^2+k\right)^2}$ = 0.9999 then determine the value of N.

Sol. **99**

$$\sum_{k=1}^{N}\frac{2k+1}{\left(k^2+k\right)^2}=\sum_{k=1}^{N}\frac{1}{k^2}-\frac{1}{(k+1)^2}=1-\frac{1}{(N+1)^2}=0.9999$$

$\Rightarrow$ N = 99

4. Let ABCD be a rectangle in which AB + BC + CD = 20 and AE = 9 where E is the mid-point of the side BC. Find the area of the rectangle.

Sol. **19**

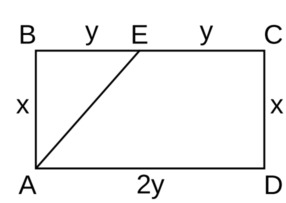
Area = 2xy

2(x + y) = 20

and x^2 + y^2 = 81

(x + y)^2 – 2xy = 81

$\Rightarrow$ 2xy = 19



5. Find the number of integer solutions to $\left||x|-2020\right|<5$.

Sol. **18**

||x| – 2020| < 5 $\Rightarrow$ 2015 < |x| < 2025

$\Rightarrow$ x = ± 2016, ± 2017, ….. ± 2024

Number of integral sol = 18

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6. What is the least positive integer by which $2^5 . 3^6 . 4^3 . 5^3 . 6^7$ should be multiplied so that, the product is a perfect square?

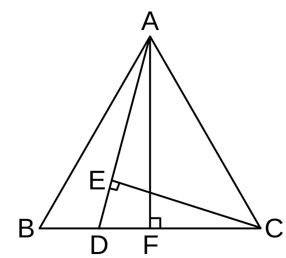
Sol. 15

Let $N = 2^5 \cdot 3^6 \cdot 4^3 \cdot 5^3 \cdot 6^7 \Rightarrow N = 2^{18} \cdot 3^{19} \cdot 5^3$

If k is the least positive integer such that k.N is perfect square, then k = 3 × 5 = 15

7. Let ABC be a triangle with AB = AC. Let D be a point on the segment BC such that BD = $48\frac{1}{61}$ and DC = 61. Let E be a point on AD such that CE is perpendicular to AD and DE = 11. Find AE.

Sol. 25



In ΔDEC

$ED^2 + CE^2 = CD^2$

$\Rightarrow CE = \sqrt{CD^2 - ED^2} = \sqrt{61^2 - 11^2}$

$\Rightarrow CE = 60$

Now let F be mid-point of BC

$\Rightarrow$ AF is perpendicular to BC as AB = AC

$$\Rightarrow BF = CF = \frac{BC}{2} = \frac{BD + DC}{2} = \frac{61 + 48 + \frac{1}{61}}{2} = 54 + \frac{31}{61}$$

Also, $DF = BF - BD = 6 + \frac{30}{61}$

In ΔAFC

$AC^2 = AF^2 + CF^2$ ..... (1)

Also, in ΔAFD

$AD^2 = DF^2 + AF^2$ ..... (2)

From equation (1) and (2), we get

$$\Rightarrow AC^2 = AD^2 + CF^2 - DF^2 = CE^2 + AE^2 \Rightarrow (AE + 11)^2 + \left(54 + \frac{31}{61}\right)^2 - \left(6 + \frac{30}{61}\right)^2 = (60)^2 + (AE)^2$$

$\Rightarrow AE = 25$

8. A 5-digit number (in base 10) has digits k, k + 1, k + 2, 3k, k + 3 in that order, from left to right. If this number is m^2 for some natural number m, find the sum of the digits of m.

Sol. 15

$m^2 = 10^4(k) + 10^3(k + 1) + 10^2(k + 2) + 10(3k) + (k + 3)$

$\Rightarrow m^2 = k(10^4 + 10^3 + 10^2 + 10 \times 3 + 1) + 10^3 + 2 \times 10^2 + 3$

$\Rightarrow m^2 = k(11131) + 1203$

For k = 1, 2, m^2 is not a perfect square

For k = 3, $m^2 = 34596 \Rightarrow m = 186$

9. Let ABC be a triangle with AB = 5, AC = 4, BC = 6. The internal angle bisector of C intersects the side AB and D. Points M and N are taken on sides BC and AC, respectively, such that DM || AC and DN || BC. If $(MN)^2 = p/q$ where p and q are relatively prime positive integers that what is the sum of the digits of |p – q|?

Sol. 2

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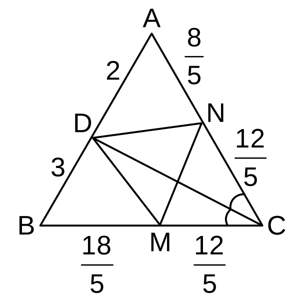
As CD is angle bisector

$\Rightarrow$ D divides AB is 4 : 6 $\Rightarrow$ AD = 2, BD = 3

AS DN is parallel to BC $\Rightarrow \frac{AD}{DB} = \frac{AN}{NC} \Rightarrow AN = \frac{8}{5},\ NC = \frac{12}{5}$

Similarly, $BM = \frac{18}{5},\ MC = \frac{12}{5}$



Now, $\cos C = \frac{4^2 + 6^2 - 5^2}{2 \times 4 \times 6} = \frac{\left(\frac{12}{5}\right)^2 + \left(\frac{12}{5}\right)^2 - MN^2}{2 \times \frac{12}{5} \times \frac{12}{5}}$

$\Rightarrow MN^2 = \frac{126}{25} = \frac{p}{q} \Rightarrow$ Sum of digits of |p – q| = 2

10. Five students take a test on which any integer score from 0 to 100 inclusive is possible. What is the largest possible difference between the median and the mean of the scores? (The median of a set of scores is the middlemost score when the data is arranged in increasing order. It is exactly the middle score when there are an odd number of scores and it is the average of the two middle scores when there are an even number of scores).

Sol. **40**

Let the scores be $\gamma - \alpha_1, \gamma - \beta_1, \gamma, \gamma + \alpha_2, \gamma + \beta_2$

$\Rightarrow$ Difference of median and mode = $\left|\frac{\alpha_2 + \beta_2 - \alpha_1 - \beta_1}{5}\right|$

Which is maximum when $\alpha_2 = \beta_2 = 100$ and $\alpha_1 = \beta_1 = 0$

$\Rightarrow$ Maximum difference = 40

11. Let X = {–5, –4, –3, –2, –1, 0, 1, 2, 3, 4, 5} and
S = {(a, b) $\in$ X $\times$ X : x^2 + ax + b and x^3 + bx + a have at least a common real zero}.
How many elements are there in S?

Sol. **24**

$\frac{x^3 + bx + a}{x^2 + ax + b} = x + \frac{a(1 - x^2)}{x^2 + ax + b}$

For atleast one root common, remainder should be zero

Case–I: a = 0, b = –1, –2, –3, –4, –5 as $a^2 \geq 4b$

Total $\rightarrow$ 6 cases

Case–II: If x = 1 $\Rightarrow$ a + b + 1 = 0, so

$\Rightarrow$ (a, b) $\in$ {(–5, 4), (–4, 3), (–3, 2), (–2, 1), (–1, 0), (4, –5), (3, –4), (2, –3), (1, –2)}

Total $\rightarrow$ 9 cases

Case–III: if x = –1 $\Rightarrow$ a – b = 1

$\Rightarrow$ (a, b) $\in$ {(5, 4), (4, 3), (3, 2), (2, 1), (1, 0), (–4, –5), (–3, –4), (–2, –3), (–1, –2)}

Total $\rightarrow$ 9 cases

So, total possible cases = 24

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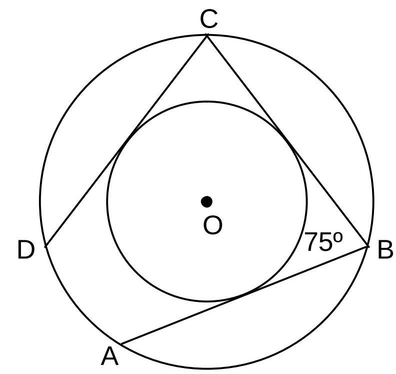
12. Given a pair of concentric circles, chords AB, BC, CD, … of the outer circle are drawn such that they all touch the inner circle. If $\angle ABC = 75^{\circ}$, how many chords can be drawn before returning to the starting point?



Sol. **24**

The angle subtended by chord at centre of circle = 105°

So, after 4 rotation point A will move $420^{\circ} = 360^{\circ} + 60^{\circ}$

So, to compensate extra 60° it has to complete 6 revolution

So, total number of chord = $4 \times 6 = 24$

13. Find the sum of all positive integers n for which $|2^n + 5^n - 65|$ is a perfect square.

Sol. **6**

$2^n + 5^n - 65 \equiv 2$ (Mod 3) if n = odd and $2^n + 5^n - 65 \equiv 0$ (Mod 3) if n = even

So, n cannot be odd

Let take n = 2k, and $2^n + 5^n - 65 = \ell^2$, ℓ is even

n = 2, n = 4 satisfies the equation

for $n \geq 6$

$\Rightarrow 2^{2k} + 5^{2k} - 65 = \ell^2$

$\ell^2 - 2^{2k} = 5^{2k} - 65$

In R.H.S last three digit is 560 and in L.H.S

$(\ell - 2^k)(\ell + 2^k)$

Last digit of ℓ is 0, 2, 4, 6, 8 and last digit of 2^k is 2, 4, 6, 8

Case–I: Last digit of ℓ = last digit of 2^k = 2, 4, 6, 8

So, possibility of last digit of L.H.S will be 40, 160, 360, 640 which are not possible

Case–II: Last digit of 2^k + last digit of ℓ = 10 $\Rightarrow$ (2, 8), (4, 6)

So, possibility of last digit of L.H.S 160, 20

So, only n = 2 and n = 4 possible

$\sum n = 6$

14. The product $55 \times 60 \times 65$ is written as the product of five distinct positive integers. What is the least possible value of the largest of these integers?

Sol. **20**

Let $M = 55 \times 60 \times 65$

$M = 2^2 \times 3 \times 5^3 \times 11 \times 13$

The least possible value of largest of integer will be greater then 13

So, $M = 5 \times 11 \times 13 \times 15 \times 20$

15 is not possible so only possibility will be 20

15. Three couples sit for a photograph in 2 rows of three people each such that no couple is sitting in the same row next to each other or in the same column one behind the order. How many arrangements are possible?

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Sol. **96**

I	II	III
IV	V	VI

Let couple are H_1w_1, H_2w_2, H_3w_3

Place 'I' can be filled in ${}^6C_1$ ways, let say H_1 is seated at place 'I'

So, w_1 have possibility at III, V, VI place

Now place 'II" can be filled in ${}^4C_1$ ways

H_1	H_2	
	w_1	

$\rightarrow {}^6C_1 \times {}^4C_1 \times \underset{\substack{\uparrow \\ \text{third couple possibilities}}}{{}^2C_1} = 48$

H_1	H_2	
		w_1

$\rightarrow {}^6C_1 \times {}^4C_1 \times \underset{\substack{\uparrow \\ \text{third couple possibilities}}}{{}^2C_1} = 48$

H_1	H_2	w_1

$\rightarrow$ This case is not possible

So, total cases = 96

16. The sides x and y of a scalene triangle satisfy $x+\frac{2\Delta}{x}=y+\frac{2\Delta}{y}$, where Δ is the area of the triangle. If x = 60, y = 63, what is the length of the largest side of the triangle?

Sol. **87**

$x+\frac{2\Delta}{x}=y+\frac{2\Delta}{y} \Rightarrow (x-y)=\frac{2\Delta(x-y)}{xy}$

$\Rightarrow \Delta=\frac{1}{2}xy$

So, angle between sides having lengths x and y is 90°

Longest side is hypotenuse having length = $\sqrt{60^2+63^2}=87$

17. How many two digit numbers have exactly 4 positive factors? (Here 1 and the number n are also considered as factors of n).

Sol. **30**

For exactly 4 positive factors, the number must be product of only 2 primes or cube of a prime number

Two digit numbers which are cubes of a prime = 1

Two digit numbers which are product of two primes = 13 + 9 + 5 + 2 = 29

Total = 30

18. If $\sum_{k=1}^{40}\left(\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}\right)=a+\frac{b}{c}$

Where a, b, c $\in$ N, b < c gcd(b, c) = 1 then what is the value of a + b?

Sol. **80**

$$\sum_{k=1}^{40}\sqrt{1+\frac{1}{k^2}+\frac{1}{(k+1)^2}}=\sum_{k=1}^{40}\sqrt{\frac{(k^4+2k^3+k^2)+(k^2+2k+1)+k^2}{k^2(k+1)^2}}=\sum_{k=1}^{40}\frac{k^2+k+1}{k(k+1)}$$

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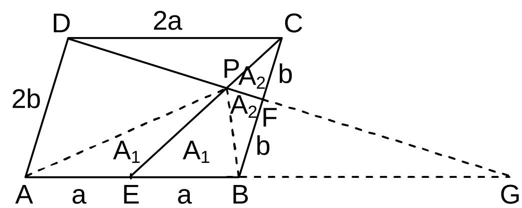
$$= \sum_{k=1}^{40} 1 + \sum_{k=1}^{40} \frac{1}{k} - \frac{1}{k+1} = 40 + 1 - \frac{1}{41} = 40 + \frac{40}{41}$$

$\Rightarrow$ a = 40 and b = 40 $\Rightarrow$ a + b = 80

19. Let ABCD be a parallelogram. Let E and F be midpoints of AB and BC respectively. The lines EC and FD intersect in P and form four triangles APB, BPC, CPD and DPA. If the area of the parallelogram is 100 sq. units, what is the maximum area in sq. units of a triangle among these four triangles?

Sol. 40



$Ar(\Delta PCD) = 50 - 2A_1$

$Ar(\Delta PAD) = 50 - 2A_2$

BF || AD and $BF = \frac{1}{2}AD$

$\Rightarrow$ BG = 2a (Midpoint Thm)

$\Delta BFG \cong \Delta CFD \Rightarrow Ar.(\Delta BFG) = 50 - 2A_1 + A_2$

$Ar(\Delta PAB) = Ar.(\Delta PBG) \Rightarrow 2A_1 = 50 - 2A_1 + 2A_2$

$\Rightarrow 2A_1 - A_2 = 25$ ..... (1)

$\Delta PEG \sim \Delta PCD \Rightarrow \sqrt{\frac{50 - 2A_1}{3A_1}} = \frac{2}{3} \Rightarrow A_1 = 15$

Using (1) $\Rightarrow A_2 = 5$

$Ar(\Delta PCD) = 50 - 2A_1 = 20$; $Ar(\Delta PAD) = 50 - 2A_2 = 40$

20. A group of women working together at the same rate can build a wall in 45 hours. When the work started, all the women did not start working together. They joined the work over a period of time, one by one, at equal intervals. Once at work, each one stayed till the work was complete. If the first woman worked 5 times as many hours as the least woman, for how many hours did the first woman work?

Sol. 75

Let these be n women. So, each women does $\frac{1}{45n}$ work/hour individually.

If T be time for which first women worked $\Rightarrow$ Total work done $= \frac{1}{45n} \frac{n}{2}\left(T + \frac{T}{5}\right) = 1 \Rightarrow$ T = 75 hrs

21. A total fixed amount of N thousand rupees is given to three persons A, B, C every year, each being given an amount proportional to her age. In the first year, A got half the total amount. When the sixth payment was made, A got six-seventh of the amount that she had in the first year; B got Rs. 1000 less than that she had in the first year; and C got twice of that she had in the first year. Find N.

Sol. 35

	A	B	C
at $t = 0$;	$\frac{N}{2}$	x	y
at $t = 5$;	$\frac{6}{7}\frac{N}{2}$	$x - 1$	$2y$
	$\frac{3N}{7}$	$\frac{3N}{7} - 2$	$\frac{N}{7} + 2$

So; $x + y = \frac{N}{2}$ ..... (1)

So; $x + 2y - 1 = \frac{4N}{7}$ ..... (2)

From equation (1) and (2), we get $y = \frac{N}{14} + 1$; $x = \frac{3N}{7} - 1$

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Since, difference in age at t = 0 and t = 5 are equal

So; $\left(\frac{N}{2}-\left(\frac{3N}{7}-1\right)\right)\lambda_1$ = difference in age

Similarly; $\left(\frac{3N}{7}-\left(\frac{3N}{7}-2\right)\right)\lambda_2$ = difference in age

So; $\frac{\left(\frac{N}{14}+1\right)}{2}=\frac{\lambda_2}{\lambda_1}$. Similarly, $\frac{\left(\frac{5N}{14}-2\right)}{\left(\frac{2N}{7}-4\right)}=\frac{\lambda_2}{\lambda_1}$

So; $\frac{N+14}{2\cdot 14}=\frac{5N-28}{2(2N-28)}$ $\Rightarrow N^2 - 196 = 35N - 28.7$

So; N = 35

22. In triangle ABC, let P and R be he feet of the perpendicular from A onto the external and internal bisectors of $\angle$ABC, respectively; and let Q and S be the feet of the perpendiculars from A onto the internal and external bisectors of $\angle$ACB, respectively. If PQ = 7, QR = 6 and RS = 8, what is the area of triangle ABC?

Sol. 84

M is the circumcentre of right angled triangle APB

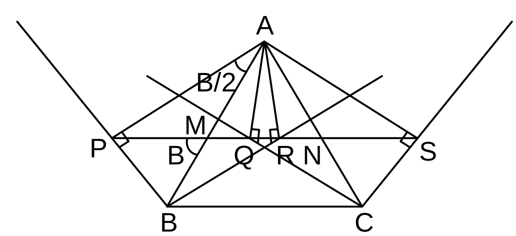
So; PM || BC, NS || BC and MN || BC

Hence, P, M, N and S are collinear

In ΔARB;

MR = MB

$\angle MRB = \angle MBR = \angle RBC = \frac{B}{2}$



So; MR || BC

Similarly, NQ || BC

Thus, P, M, Q, R, N, S are collinear

$\Rightarrow PM = MR = \frac{13}{2}$, QN = NS = 7 $\Rightarrow$ AB = 13, AC = 14, MN = $\frac{15}{2}$ $\Rightarrow$ BC = 15

So; area = 84. Let AN = MB, AN = NC

23. The incircle Γ of a scalene triangle ABC touches BC at D, CA at E and AB at F. Let r_A be the radius of the circle inside ABC which is tangent of Γ and the sides AB and AC. Define r_B and r_C similarly. If $r_A = 16$, $r_B = 25$ and $r_C = 36$. determine the radius of Γ.

Sol. 74

Circle of radius r is ex-circle for the remaining circles.

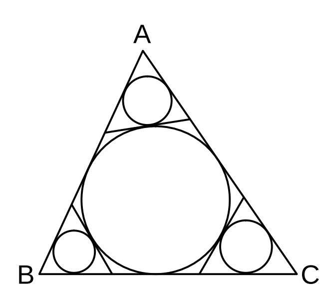
So, from the property of ex-circles

$r = \sqrt{r_A r_B} + \sqrt{r_B r_C} + \sqrt{r_C r_A}$

$= \sqrt{16.25} + \sqrt{25.36} + \sqrt{36.16}$

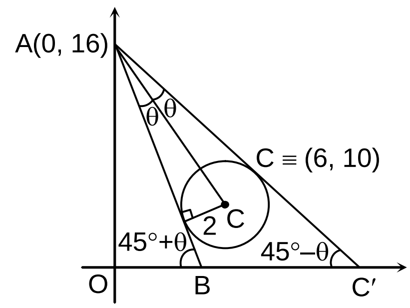
= 20 + 30 + 24

r = 74



24. A light source at the point (0, 16) in the coordinate plane casts light in all directions. A disc (a circle along with its interior) of radius 2 with centre at (6, 10) casts a shadow on the X-axis. The length of the shadow can be written in the form $m\sqrt{n}$ where m, n are positive integers and n is square-free. Find m + n

Sol. **21**



$\tan(45^\circ + \theta) = \dfrac{16}{OB}$

$\Rightarrow OB = \dfrac{16}{\tan(45^\circ + \theta)}$ ..... (1)

Similarly, $OC = \dfrac{16}{\tan(45^\circ - \theta)}$ ..... (2)

$\because AC = \sqrt{36 + 36} = 6\sqrt{2}$

So, $\tan\theta = \dfrac{2}{2\sqrt{17}} = \dfrac{1}{\sqrt{17}}$

Hence, length of shadow = BC = OC – OB = $16\left(\dfrac{\sqrt{17}+1}{\sqrt{17}-1} - \dfrac{\sqrt{17}-1}{\sqrt{17}+1}\right)$ = $4\sqrt{17}$

So, m + n = 21

25. For a positive integer n let $\langle n \rangle$ denote the perfect square integer closest to n. For example, $\langle 74 \rangle = 81, \langle 18 \rangle = 16$. If N is the smallest positive integer such that

$$\langle 91 \rangle \cdot \langle 120 \rangle \cdot \langle 143 \rangle \cdot \langle 180 \rangle \cdot \langle N \rangle = 91 \cdot 120 \cdot 143 \cdot 180 \cdot N$$

Find the sum of the square of the digits of N.

Sol. **56**

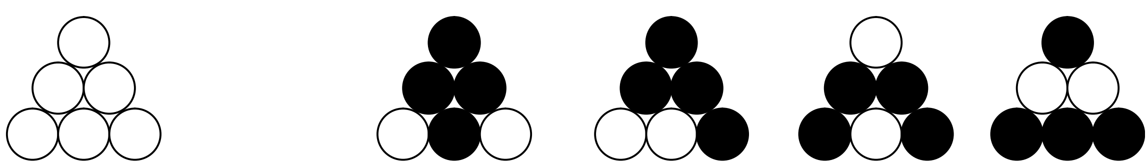
$10^2 \cdot 11^2 \cdot 12^2 \cdot 13^2 \cdot \langle N \rangle = (13 \cdot 7)(12 \cdot 10)(13 \cdot 11)(18 \cdot 10)N$

$\Rightarrow 22\langle N \rangle = (21)N$

Smallest N should be 22·21 in this case $\langle N \rangle$ = 441 = 21·21

Hence, L.H.S = R.H.S

26. In the figure below, 4 of the 6 disks are to be colored black and 2 are to be colored white. Two colorings that can be obtained from one another by a rotation or a reflection of the entire figure are considered the same.



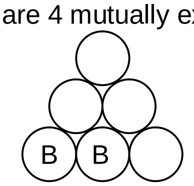
(i) (ii) (iii) (iv)

There are only four such colorings for the given two colors, as shown in figure. In how many ways can we color the 6 disks such that 2 are colored black, 2 are colored white, 2 are colored blue with the given identification condition?

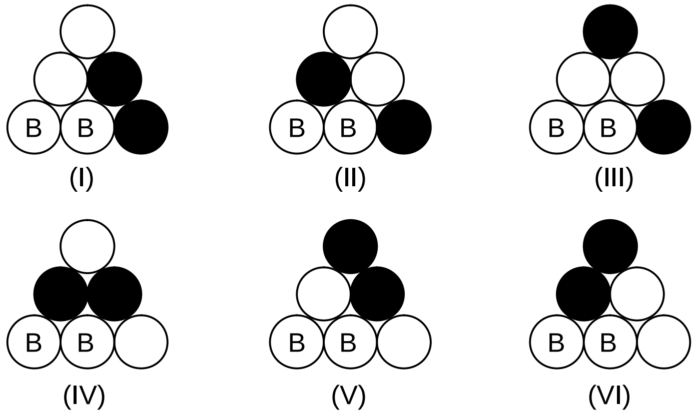
Sol. **18**

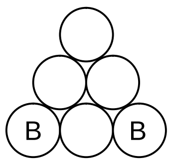
There are 4 mutually exclusive ways of colouring two disc with blue colour

(i)

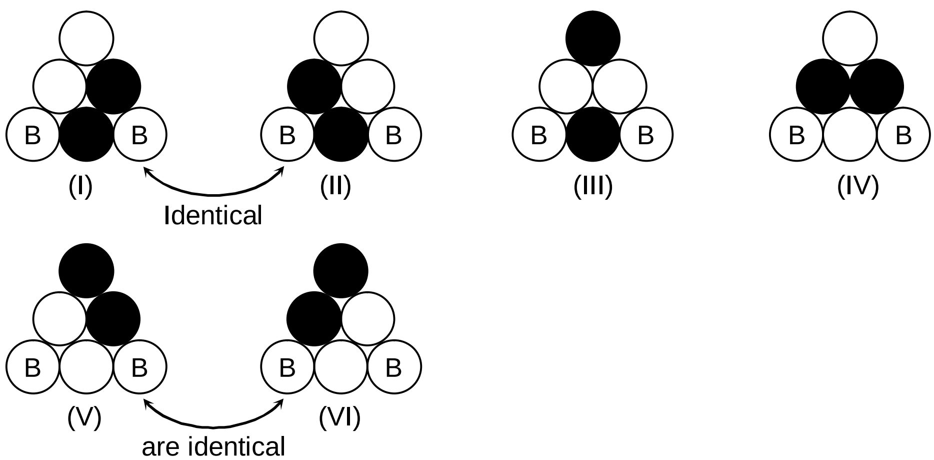


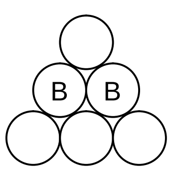
$\longrightarrow$ In this case there are ${}^4C_2$ ways to colour rest 4 disc



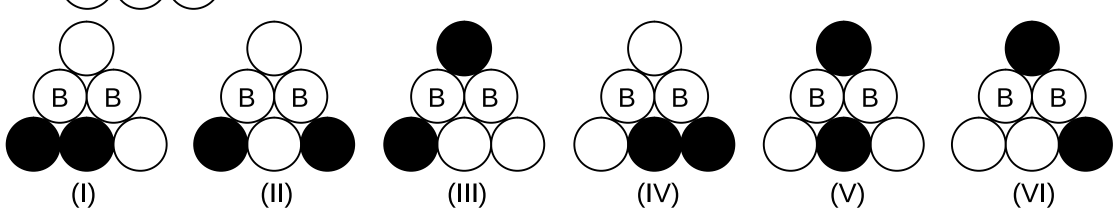
(ii) 

⟶ There are only four ways to colour as out of six I and II are identical as well as V and VI are identical

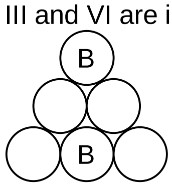


(iii) 

⟶ There are 4 ways to colour rest 4 disc



III and VI are identical and I and IV are identical

(iv) 

⟶ There are only 4 ways to colour rest 4 disc

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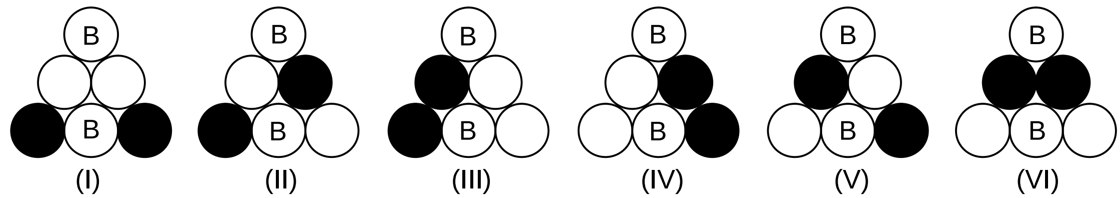
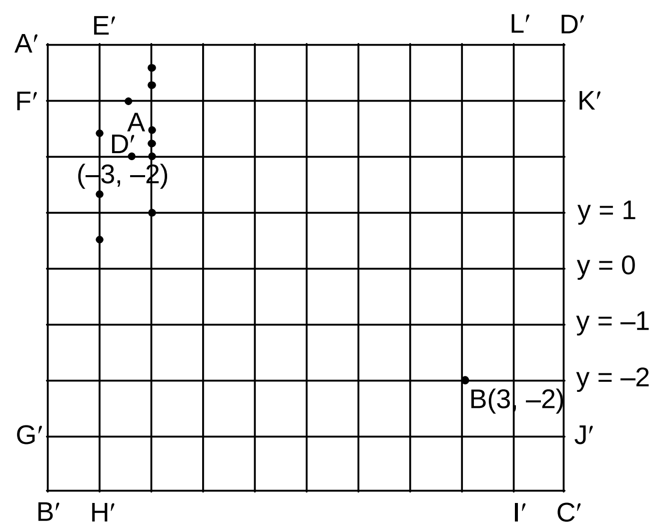


III and VI are identical and II and V are identical
So, total ways of colouring are $6 + 3 \times 4 = 18$

27. A bug travels in the coordinate plane moving only along the lines that parallel to the x-axis or y-axis. Let A(–3, 2) and B(3, –2). Consider all possible paths of the bug from A and B of length at most 14. How many points with integer coordinates lie on at least one of these paths?

Sol. **87**



With given condition bug can use all points except $A', F', E', B', H', G', I', C', J', K', L', D'$ in his path
So, total number of points in atleast one of the path will be $= 11 \times 9 - 4 \times 3 = 87$

28. A natural number n is said to be good if n is the sum of r consecutive positive integers, for some $r \geq 2$. Find the number of good numbers in the set {1, 2, …. 100}.

Sol. **93**

Number of ways of writing n as	**new cases**
Sum of 2 consecutive number	49
Sum of 3 consecutive number	16
Sum of 4 consecutive number	16
Sum of 5 consecutive number	4
Sum of 6 consecutive number	0
Sum of 7 consecutive number	2
Sum of 8 consecutive number	5
Sum of 9 consecutive number	0
Sum of 10 consecutive number	0
Sum of 11 consecutive number	1
Sum of 12 consecutive number	0
Sum of 13 consecutive number	0
Total :	**93**

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29. Positive integers a, b, c satisfy $\frac{ab}{a-b} = c$. What is the largest possible value of a + b + c not exceeding 99?

Sol. **99**

Given a, b, c $\in I^+$

$\frac{ab}{a-b} = c \Rightarrow b = \frac{ac}{a+c}$ if a = km, c = kn, then b = kr

$\frac{1}{b} = \frac{1}{a} + \frac{1}{c}$ ..... (1)

From equation (i), we get

$\frac{1}{kr} = \frac{1}{km} + \frac{1}{kn} \Rightarrow \frac{1}{r} = \frac{1}{m} + \frac{1}{n}$ ..... (2)

m = 3, n = 6, r = 2 is one of the solution of the equation (2)

So, a = 3k, c = 6k, b = 2k

Given a + b + c $\le$ 99 ; (3 + 6 + 2)k $\le$ 99 ; k $\le$ 9

So, a = 27, c = 54 and b = 18

a + b + c = 27 + 54 + 18 = 99

30. Find the number of pairs (a, b) of natural numbers such that b is a 3-digit number, a + 1 divides b – 1 and b divides $a^2 + a + 2$.

Sol. **16**

Given a, b $\in I^+$ is a 3 digit number

a + 1 divides b – 1

So, b – 1 = k(a + 1) ; b = k(a + 1) + 1 and b divides $a^2 + a + 2$

k(a + 1) + 1 divides $a^2 + a + 2 \Rightarrow$ k(a + 1) + 1 divides $k(a^2 + a + 2) - a(k(a + 1) + 1))$

k(a + 1) + 1 divides 2k – a

ka + k + 1 / 2k – a

$\Rightarrow$ a = 2k

ka + k + 1 / 0

b = k(2k + 1) + 1

= $2k^2 + k + 1$ as b is 3 digit number

$100 \le 2k^2 + k + 1 \le 999$

$\sqrt{\frac{99}{2} + \frac{1}{16}} \le k + \frac{1}{4} \le \sqrt{\frac{998}{2} + \frac{1}{16}}$

60 ... $\le$ k $\le$ 220 ...

k $\in$ [7, 22]

So, total comes = 16

END OF THE PAPER

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