FIITJEE
LIMITED

33^{rd} Indian National Mathematical Olympiad-2018

Time : 4 hours January 21, 2018

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- All questions carry equal marks. Maximum marks : 102
- Answer all the questions.
- Answers to each question should start on a new page. Clearly indicate the question number.

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1. Let ABC be a non-equilateral triangle with integer sides. Let D and E be respectively the mid-points BC and CA ; let G be the centroid of triangle ABC. Suppose D, C, E, G are concyclic. Find the least possible perimeter of triangle ABC.

Sol. Given AG·AD = AE · AC

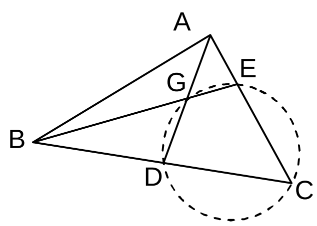
$\Rightarrow a^2 + b^2 = 2c^2$

$\Rightarrow \left(\frac{a-b}{2}\right)^2 + \left(\frac{a+b}{2}\right)^2 = c^2$

For least parameter of the ΔABC

$\Rightarrow a - b = 10,\ a + b = 24$ and $c = 13$

$\Rightarrow a = 10,\ b = 7$ and $c = 13$



2. For any natural number n consider a $1 \times n$ rectangular board made up of n unit squares. This is covered by three types of tiles : 1×1 red tile, 1×1 green tile and 1×2 blue domino. (For example, we can have 5 types of tiling when n = 2 ; red-red; red-green; green-red; green-green; and blue). Let t_n denote the number of ways of covering $1 \times n$ rectangular board by these types of tiles. Prove that t_n divides t_{2n+1}.

Sol. Consider the $(n+1)^{th}$ tile

Case 1 : It is red or green –

number of ways = $2.\ t_n^2$

Case 2 : it is blue –

we can have 2 configurations

[$(n-1)$ tiles] [blue] [n tiles] or [n tiles] [blue tile] [$(n-1)$ tiles]

number of ways = $2.t_n\, t_{n-1}$.

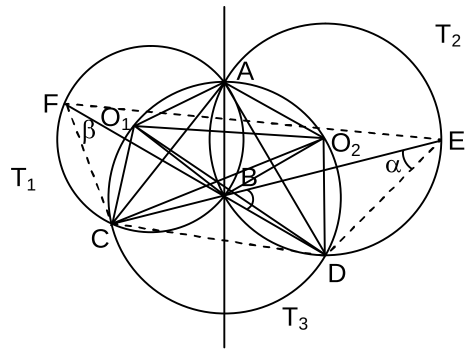
$\therefore$ total number of ways = $t_{2n+1} = 2t_n^2 + 2t_n t_{n-1}$

$= 2t_n\,(t_n + t_{n-1})$

$\Rightarrow$ t_n divides t_{2n+1}

3. Let τ_1 and τ_2 be two circles with respective centres O_1 and O_2 intersecting in two distinct points A and B such that $\angle O_1AO_2$ is an obtuse angle. Let the circumcircle of triangle O_1AO_2 intersect τ_1 and τ_2 respectively in points C($\neq$ A) and D($\neq$ A). Let the line CB intersect τ_2 in E let the line DB intersect τ_1 in F. Prove that the points C, D, E, F are concyclic.

Sol. Let $\angle CED = \alpha \therefore \angle DAB = \alpha$
$\angle CFB = \beta \therefore \angle CAB = \beta$
$\therefore \angle CO_1B = 2\beta$, $\angle DO_2B = 2\alpha$
$\angle CAD = \alpha + \beta = \angle CO_1D$
$\therefore \angle BO_1D = \alpha - \beta$
Similarly $\angle CO_2B = \beta - \alpha$
$\Rightarrow \alpha = \beta$ hence proved

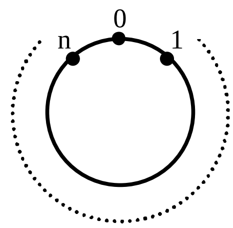


4. Find all polynomials with real coefficients P(x) such that $P(x^2 + x + 1)$ divides $P(x^3 - 1)$.

Sol. Let $P(x^3 - 1) = a_0(x^3 - 1)^n + a_1(x^3 - 1)^{n-1} + ... + a_n$
$P(x^2 + x + 1) = a_0(x^2 + x + 1)^n + a_1(x^2 + x + 1)^{n-1} + ... + a_n$
and let $q(x) = (b_0 x^n + b_1 x^{n-1} + ... + b_n)$
Such that $P(x^3 - 1) = P(x^2 + x + 1)q(x)$
Equating the coefficients, we get
$a_1 = a_2 = ... = a_n = 0$
$\Rightarrow P(x^3 - 1) = a_0(x^3 - 1)^n$
$\Rightarrow P(x) = ax^n$

5. There are $n \geq 3$ girls in a class sitting around a circular table, each having some apples with her. Every time the teacher notices a girl having more apples than both of her neighbours combined, the teacher takes away one apple from that girl and gives one apple each to her neighbours. Prove that this process stops after a finite number of steps. (Assume that the teacher has an abundant supply of a apples).

Sol. a_I is number of apples with i^{th} girl
consider $S = |a_0 - a_1| + |a_1 - a_2| + ...+ |a_n - a_n| + |a_n - a_0|$
If i^{th} the girl is touched by this operation then
$|a_{i-2} - a_{i-1}|$ can change by ± 1
$|a_{i+1} - a_{i+2}|$ can change by ± 1
$|a_{i-1} - a_i|$ and $|a_i - a_{i+1}|$ both decreases by 2;
(since a_i is greater than a_{i-1} and a_{i+1})
So, S decreases with every operation
But $S \geq 0 \Rightarrow$ process has to end in finite number of steps.



6. Let N denote the set of all natural numbers and let $f : N \to N$ be a function such that
(a) $f(mn) = f(m) f(n)$ for all m, n in N.
(b) $m + n$ divides $f(m) + f(n)$ for all m, n in N.
Prove that there exists an odd natural number k such that $f(n) = n^k$ for all n in N.

Sol. $f(n) = n^k$
$f(mn) = (mn)^k = m^k n^k$
Also, $f(m) + f(n) = m^k + n^k$
$\because\ m^k + n^k = (m + n)(m^{k-1} - m^{k-2}n + m^{k-3}n^2 - ..... + n^{k-1})$; where k is odd integer

$$= (m+n)m^{k-1}\left(\frac{1-\left(\frac{-n}{m}\right)^k}{1-\left(-\frac{n}{m}\right)}\right)$$

$$= (m+n)m^{k-1}\frac{\left(\frac{m^k+n^k}{m^k}\right)}{\left(\frac{m+n}{m}\right)}$$

$= m^k + n^k$

***** **End** *****

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