Regional Mathematical Olympiad – 2017

Time : 3 hours **October 08, 2017**

Instructions:

- Calculators (in any form) and protractors are not allowed.
- Rulers and compasses are allowed.
- Answer all the questions.
- All questions carry equal marks. Maximum marks : 102
- Answer to each question should start on a new page. Clearly indicate the question number.

1. Let AOB be a given angle less than 180^o and let P be an interior point of the angular region determined by $\angle AOB$. Show, with proof, how to construct, using only ruler and compasses, a line segment CD passing through P such that C lies on the ray OA and D lies on the ray OB, and CP : PD = 1 : 2.

2. Show that the equation $a^3+(a+1)^3+(a+2)^3+(a+4)^3+(a+5)^3+(a+6)^3=b^4+(b+1)^4$ has no solutions in integers a, b.

3. Let $P(x)=x^2+\frac{1}{2}x+b$ and $Q(x)=x^2+cx+d$ be two polynomials with real coefficients such that $P(x)Q(x)=Q(P(x))$ for real x. Find all the real roots of P (Q(X)) = 0.

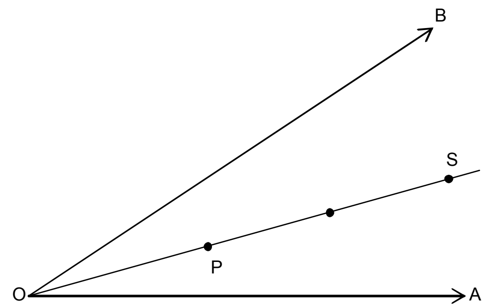
4. Consider n^2 unit squares in the xy-plane centered at point (i, j) with integer coordinates, $1 \leq i \leq n$, $1 \leq j \leq n$. It is required to colour each unit square in such a way that when ever $1 \leq i < j \leq n$ and $1 \leq k < \ell \leq n$, the three squares with centers at (i, k), (j, k), (j, ℓ) have distinct colours. What is the lest possible number of colours needed?

5. Let Ω be a circle with a chord AB which is not a diameter. Let Γ_1 be a circle on one side of AB such that it is tangent to AB at C and internally tangent to Ω at D. Likewise, let Γ_2 be a circle on the other side of AB such that it is tangent to AB at E and internally tangent to Ω at F. Suppose the line DC intersects Ω at $X \neq D$ and the line FE intersects Ω at $Y \neq F$. Prove that XY is a diameter of Ω.

6. Let x, y, z be real numbers, each greater than 1. Prove that
$$\frac{x+1}{y+1}+\frac{y+1}{z+1}+\frac{z+1}{x+1} \leq \frac{x-1}{y-1}+\frac{y-1}{z-1}+\frac{z-1}{x-1}$$

Solutions

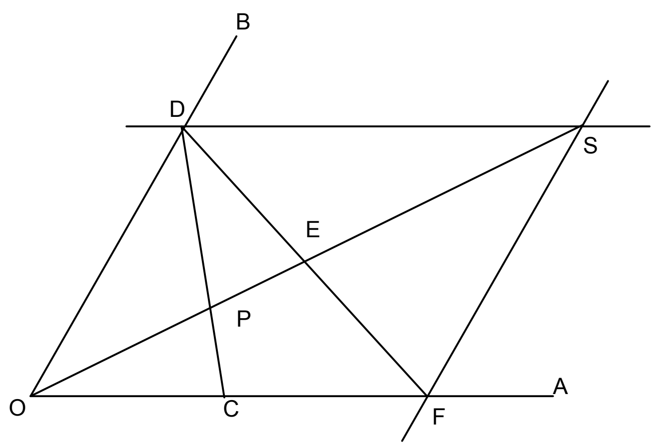
1.



Step 1 : Join OP and extend the ray OP

Step 2 : Mark a point S on the ray OP at a distance 3OP.

Step 3 : Draw a line $||^{\ell}$ to ray OA from point S and a line $||^{\ell}$ to ray OB from point S.



Let the lines intersect OA at F and OB at D.
Intersection of OS and DF is midpoint of DF.
In ΔODF, E is midpoint

$\Rightarrow$ OE is median and since $\frac{OP}{OE} = \frac{2}{1} \Rightarrow$ P is centroid.

$\therefore$ Join D and P to intersect OA at C

$\Rightarrow$ CD is median and $\frac{CP}{PD} = \frac{1}{2}$

2. Given equation simplifies to $7\left[a^3 + 9a^2 + 39a + 63\right] = b^4 + (b+1)^4$

LHS is divisible by 7 but RHS is never divisible by 7 which can be checked easily by putting $b = 7k-3,\ 7k-2,\ 7k-1,\ 7k, 7k+1, 7k+2, 7k+3$

Hence no solution.

3. $\left(x^2 + \frac{x}{2} + b\right)\left(x^2 + cx + d\right) = \left(x^2 + \frac{x}{2} + b\right)^2 + c\left(x^2 + \frac{x}{2} + b\right) + d$

Comparing we get $c = \frac{1}{2}, b = -\frac{1}{2}, d = 0$

$$P(Q(x)) = \left(x^2 + \frac{x}{2} - \frac{1}{2}\right)\left(x^2 + \frac{x}{2}\right)$$

Its roots are $-1, \frac{1}{2}$

4.

V	N_1	N_2	N_3	N_4	N_5	N_6
I	V	N_1	N_2	N_3	N_4	N_5
P	I	V	N_1	N_2	N_3	N_4
O	P	I	V	N_1	N_2	N_3
Y	O	P	I	V	N_1	N_2
B	Y	O	P	I	V	N_1
R	B	Y	O	P	I	V

Number of colors are $= n + n - 1 = 2n - 1$

5. Let $\angle DCB = \alpha \Rightarrow \angle CO_1D = 2\alpha$

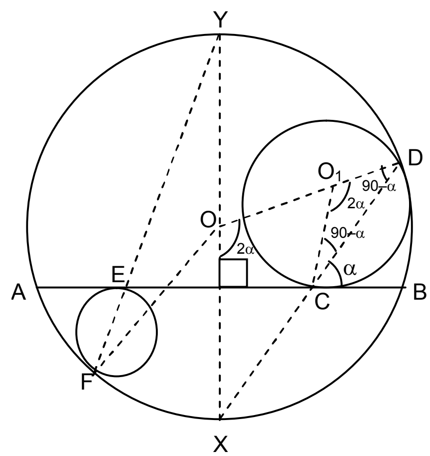
$\Rightarrow \angle XOD = 2\alpha$

$\Rightarrow OY \,/\!/\, O_1C$

So, $OX \perp AB$

Similarly OY is $\perp$ to AB

OY and OX both $\perp$r to AB and passing through O so XY is diameter



6.
$$\left(\frac{x+1}{y+1} - \frac{x-1}{y-1}\right) + \left(\frac{y+1}{z+1} - \frac{y-1}{z-1}\right) + \left(\frac{z+1}{x+1} - \frac{z-1}{x-1}\right)$$

$$= \frac{2(y-x)}{y^2-1} + \frac{2(z-y)}{z^2-1} + \frac{2(x-z)}{x^2-1}$$

Let $x > y > z$

$\therefore\ y - x < 0, z - y < 0$ and $x - z > 0$

$|y-x| + |z-y| = |z-x|$

Now $x^2 - 1 > y^2 - 1$

$x^2 - 1 > z^2 - 1$

$\therefore \left|\frac{2(x-z)}{x^2-1}\right|$ is least

Hence $\frac{2(y-x)}{y^2-1} + 2\frac{(z-y)}{z^2-1} + \frac{2(x-z)}{x^2-1} \le 0$